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Lower bounds for the eigenvalue of the transversal Dirac operator on a Kähler foliation

Seoung Dal Jung [a,*], Tae Ho Kang [b]

[a] Department of Mathematics, Jheju National University, Cheju 690-756, South Korea
[b] Department of Mathematics, University of Ulsan, Ulsan 680-749, South Korea

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Abstract

On a foliated Riemannian manifold with a Kähler spin foliation, we give a lower bound for the square of the eigenvalues of the transversal Dirac operator. We prove, in the limiting case, that the foliation is a minimal, transversally Einsteinian of odd complex dimension with nonnegative constant transversal scalar curvature.

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1. Introduction

On a foliated Riemannian manifold $(M, g_M, \mathcal{F})$ with a transverse spin structure, it was shown by Jung [3] that for any eigenvalue λ of the transversal Dirac operator D_{tr}, the estimation

$$\lambda^2 \geq \frac{q}{4(q-1)} K_\sigma^0 \tag{1.1}$$

holds, where $q = \operatorname{codim} \mathcal{F}$, $K_\sigma^0 = \min(\sigma^\nabla + |\kappa|^2)\ (\geq 0)$. Here σ^∇ is a transversal scalar curvature and κ is the mean curvature form of $\mathcal{F}$. In the limiting case, the foliation is a minimal transversally Einsteinian with constant transversal scalar curvature. The essential point in the proof of (1.1) was the introduction of a modified connection of the form

$$\nabla_X^f \phi = \nabla_X \phi + f\pi(X) \cdot \phi,$$

* Corresponding author.
E-mail addresses: sdjung@cheju.ac.kr (S.D. Jung), thkang@mail.ulsan.ac.kr (T.H. Kang).

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where f is a real valued basic function and $\pi : TM \to Q$ is a projection from the tangent bundle onto the normal bundle Q (see Section 2). In the case that the equality in (1.1) holds, the eigenspinor ϕ_1 corresponding to the first eigenvalue λ_1 with $\lambda_1^2 = (q/4(q-1))K_\sigma^0$ is a transversal Killing spinor, i.e., $\nabla_X^{f_1}\phi = 0$, $f_1 = \lambda_1/q$ and the foliation $\mathcal{F}$ is minimal. Hence we can prove that the equality in (1.1) on the Kähler spin foliation of $q \neq 2$ is not possible. Namely, if one takes the basic 2-form Ω as an endomorphism of the foliated spinor bundle, then since $\mathcal{F}$ is minimal, one obtains the equation

$$D_{\mathrm{tr}}(\Omega\phi_1) = \frac{q-4}{q}\lambda_1\Omega\phi_1. \tag{1.2}$$

Since the number $((q-4)/q)\lambda_1$ cannot be an eigenvalue of D_{tr} for $q \neq 2$, (1.2) implies $\Omega\phi = 0$. Hence by straight calculation, it can be shown that $D_{\mathrm{tr}}\phi_1 = \lambda_1\phi_1$ and $\Omega\phi_1 = 0$ imply $\phi_1 = 0$, which implies that in the Kähler spin foliation, the equality in (1.1) does not hold. Hence we obtain a better lower bound for the eigenvalues of D_{tr} than the one in (1.1). Namely, we prove the following theorem.

Main Theorem. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a Kähler spin foliation $\mathcal{F}$ of codimension $q = 2n$ and a bundle-like metric g_M with $\kappa \in \Omega_B^1(\mathcal{F})$. Assume that the mean curvature κ of $\mathcal{F}$ satisfies $\delta\kappa = 0$ and transversally holomorphic. If $K_\sigma = \sigma^\nabla + |\kappa|^2 \geq 0$, then the eigenvalue λ of D_{tr} satisfies*

$$\lambda^2 \geq \frac{q+2}{4q}K_\sigma^0, \tag{1.3}$$

where $K_\sigma^0 = \min K_\sigma$. If $(1/2)\sqrt{((q+2)/q)K_\sigma^0}$ itself is an eigenvalue of D_{tr}, then the Kähler foliation $\mathcal{F}$ is a minimal, transversally Einsteinian of odd complex dimension n with nonnegative constant transversal scalar curvature σ^∇.

Main Theorem is a generalization of the one on an ordinary Kähler spin manifold by Kirchberg [5]. Namely, on the closed Kähler spin manifold M^{2n} with positive scalar curvature R, the eigenvalues λ of the Dirac operator D satisfies the following:

$$\lambda^2 \geq \frac{m+2}{4m}R_0, \quad m = 2n, \tag{1.4}$$

where $R_0 = \min R$. In the limiting case, the manifold is an Einstein of odd complex dimension m.

This paper is organized as follows. In Section 2, we give the definition of a Kähler foliation. In Section 3, we review the transversal spin structure on the Riemannian foliation and modify many properties of Kirchberg's paper [5] for foliation. In Section 4, we study some basic properties of the transversal Dirac operator. In Section 5, we give a lower bound for the square of the eigenvalues of the transversal Dirac operator. In Section 6, we prove, in the limiting case, that the foliation is a minimal, transversally Einsteinian of odd complex dimension with nonnegative constant transversal scalar curvature.

This paper is based on [5]. Since the techniques are similar to those in [5], we omit proofs of many equations except for equations related to the mean curvature form κ of the foliation $\mathcal{F}$.

2. Kähler foliation

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$.

We recall the exact sequence

$$0 \to L \to TM \stackrel{\pi}{\to} Q \to 0$$

determined by the tangent bundle L and the normal bundle Q of $\mathcal{F}$. The assumption of g_M to be a bundle-like metric means that the induced metric g_Q on the normal bundle $Q \cong L^\perp$ satisfies the holonomy invariance condition $\theta(X)g_Q = 0$ for all $X \in \Gamma L$, where $\theta(X)$ denotes the Lie derivative with respect to X.

For a distinguished chart $\mathcal{U} \subset M$ the leaves of $\mathcal{F}$ in $\mathcal{U}$ are given as the fibers of a Riemannian submersion $f : \mathcal{U} \to \mathcal{V} \subset N$ onto an open subset $\mathcal{V}$ of a model Riemannian manifold N. For overlapping charts $U_\alpha \cap U_\beta$, the corresponding local transition functions $\gamma_{\alpha\beta} = f_\alpha \circ f_\beta^{-1}$ on N are isometries. Further, we denote by ∇ the canonical connection of the normal bundle $Q = TM/L$ of $\mathcal{F}$. It is defined by

$$\nabla_X s = \pi([X, Y_s]) \quad \text{for} \quad X \in \Gamma L, \qquad \nabla_X s = \pi(\nabla^M_X Y_s) \quad \text{for} \quad X \in \Gamma L^\perp, \tag{2.1}$$

where $s \in \Gamma Q$, and $Y_s \in \Gamma L^\perp$ corresponding to s under the canonical isomorphism $L^\perp \cong Q$. The connection ∇ is metric and torsion free. It corresponds to the Riemannian connection of the model space N [4]. The curvature R^∇ of ∇ is defined by

$$R^\nabla_{XY} = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \quad \text{for} \quad X, Y \in TM.$$

Since $\mathrm{i}(X)R^\nabla = 0$ for any $X \in \Gamma L$ [4], we can define the (transversal) Ricci curvature $\rho^\nabla : \Gamma Q \to \Gamma Q$ and the (transversal) scalar curvature σ^∇ of $\mathcal{F}$ by

$$\rho^\nabla(s) = \sum_a R^\nabla_{sE_a} E_a, \qquad \sigma^\nabla = \sum_\alpha g_Q(\rho^\nabla(E_a), E_a),$$

where $\{E_a\}_{a=1,\dots,q}$ is an orthonormal basis for Q. The foliation $\mathcal{F}$ is said to be (transversally) *Einsteinian* if the model space N is Einsteinian, that is,

$$\rho^\nabla = \frac{1}{q}\sigma^\nabla \cdot \mathrm{id} \tag{2.2}$$

with constant transversal scalar curvature σ^∇. The *second fundamental form* of α of $\mathcal{F}$ is given by

$$\alpha(X, Y) = \pi(\nabla^M_X Y) \quad \text{for} \quad X, Y \in \Gamma L. \tag{2.3}$$

It is trivial that α is Q-valued, bilinear and symmetric. The *mean curvature vector field* of $\mathcal{F}$ is then defined by

$$\tau = \sum_i \alpha(E_i, E_i), \tag{2.4}$$

where $\{E_i\}_{i=1,\dots,p}$ is an orthonormal basis of L. The dual form κ, the *mean curvature form* for L, is then given by

$$\kappa(X) = g_Q(\tau, X) \quad \text{for} \quad X \in \Gamma Q. \tag{2.5}$$

The foliation $\mathcal{F}$ is said to be *minimal* (or *harmonic*) if $\kappa = 0$.

Let $\Omega_B^r(\mathcal{F})$ be the space of all *basic r-forms*, i.e.,

$$\Omega_B^r(\mathcal{F}) = \{\phi \in \Omega^r(M) | \mathrm{i}(X)\phi = 0, \theta(X)\phi = 0 \;\text{ for }\; X \in \Gamma L\}.$$

The foliation $\mathcal{F}$ is said to be *isoparametric* if $\kappa \in \Omega_B^1(\mathcal{F})$. We already know that κ is closed, i.e., $\mathrm{d}\kappa = 0$ if $\mathcal{F}$ is isoparametric [8]. Since the exterior derivative preserves the basic forms (that is, $\theta(X)\,\mathrm{d}\phi = 0$ and $\mathrm{i}(X)\,\mathrm{d}\phi = 0$ for $\phi \in \Omega_B^r(\mathcal{F})$), the restriction $d_B = d|_{\Omega_B^*(\mathcal{F})}$ is well defined. Let δ_B be the adjoint operator of d_B. Then it is well known [3] that

$$d_B = \sum_a \theta^a \wedge \nabla_{E_a}, \qquad \delta_B = -\sum_a \mathrm{i}(E_a)\nabla_{E_a} + \mathrm{i}(\kappa_B), \tag{2.6}$$

where $\{E_a\}_{a=1,\dots,q}$ is a local orthonormal basic frame in Q and $\{\theta^a\}$ its g_Q-dual 1-form. The *basic Laplacian* acting on $\Omega_B^*(\mathcal{F})$ is defined by

$$\Delta_B = d_B\delta_B + \delta_B d_B. \tag{2.7}$$

If $\mathcal{F}$ is the foliation by points of M, the basic Laplacian is the ordinary Laplacian.

Further, $\mathcal{F}$ is said to be a *Kähler foliation* [7] if it is modeled on a Kähler manifold. Namely, by a Kähler foliation $\mathcal{F}$ we mean a foliation satisfying the following conditions: (i) $\mathcal{F}$ is Riemannian, with a bundle-like metric g_M on M inducing the holonomy invariant metric g_Q on $Q \equiv L^\perp$, (ii) there is a holonomy invariant almost complex structure $J : Q \to Q$, where $\dim Q = q\ (= 2n)$ (real dimension), with respect to which g_Q is Hermitian, i.e.,

$$g_Q(JX, JY) = g_Q(X, Y) \tag{2.8}$$

for $X, Y \in \Gamma Q$, and (iii) if ∇ is almost complex, i.e., $\nabla J = 0$. Note that

$$\Omega(X, Y) = g_Q(X, JY) \tag{2.9}$$

defines a basic 2-form Ω, which is closed as a consequence of $\nabla g_Q = 0$ and $\nabla J = 0$. Then we can express the basic 2-form Ω by

$$\Omega = \sum_{k=1}^{n} \theta^{2k-1} \wedge \theta^{2k}. \tag{2.10}$$

For a Kähler foliation, we have the following identities [7]:

$$R^\nabla_{XY} J = J R^\nabla_{XY}, \qquad R^\nabla_{JXJY} = R^\nabla_{XY}, \tag{2.11}$$

$$R^\nabla_{XY}Z + R^\nabla_{YZ}X + R^\nabla_{ZX}Y = 0, \tag{2.12}$$

where X, Y and Z are elements of ΓQ. In the sequal it will be convenient to use the following orthonormal frame on M. For $x \in M$, let $\{e_A\}_{A=1,\dots,p+q}$ be an oriented orthonormal basis of T_xM with $\{e_i\}_{i=1,\dots,p}$ in L_x and $\{e_\alpha, Je_\alpha\}_{\alpha=p+1,\dots,p+n}$ in $L_x^\perp$ ($\mathcal{F}$ is of codimension $q = 2n$

on M^{p+q}). The transversal Kähler property of $\mathcal{F}$ allows then to extend e_α, Je_α to local vector fields $E_\alpha, JE_\alpha \in \Gamma L^\perp$ such that

$$(\nabla_{E_\alpha} E_\beta)_x = 0, \qquad (\nabla_{E_\alpha} JE_\beta)_x = 0,$$
$$(\nabla_{JE_\alpha} E_\beta)_x = 0, \qquad (\nabla_{JE_\alpha} JE_\beta)_x = 0. \tag{2.13}$$

As a consequence of torsion freeness [4]

$$[E_\alpha, E_\beta]_x, \qquad [E_\alpha, JE_\beta]_x, \qquad [JE_\alpha, JE_\beta]_x \in L_x. \tag{2.14}$$

The E_α, JE_α can be chosen as (local) infinitesimal automorphisms of $\mathcal{F}$, so that

$$\nabla_X E_\alpha = \pi[X, E_\alpha] = 0 \quad \text{for} \quad X \in \Gamma L. \tag{2.15}$$

We can complete E_α, JE_α by the Gram–Schmidt process to a moving local frame by adding $E_i \in \Gamma L$ with $(E_i)_x = e_i$.

An infinitesimal automorphism Y gives rise to a *transversally holomorphic field* $s = \pi(Y)$ if and only if

$$\theta(Y)J = 0, \tag{2.16}$$

where for $Z \in \Gamma L^\perp$, $(\theta(Y)J)(Z) = \theta(Y)(JZ) - J(\theta(Y)Z)$. But this expression equals $\pi[Y, JZ] - J\pi[Y, Z]$, which yields the formula

$$(\theta(Y)J)(Z) = -\nabla_{JZ}s + J\nabla_Z s,$$

so that (2.16) holds if and only if

$$\nabla_{JZ}s = J\nabla_Z s \quad \text{for all} \quad Z \in \Gamma L^\perp. \tag{2.17}$$

3. The structures of the foliated spinor bundle of a Kähler spin foliation

In this section, we shall modify all the definitions and notations of Kirchberg's paper [5] for foliation. We first define the Kähler spin foliation. Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a Kähler foliation $\mathcal{F}$ of codimension $q = 2n$ and a bundle-like metric g_M with respect to $\mathcal{F}$. Let $\mathrm{SO}(q) \to P_{\mathrm{so}} \to M$ be the principal bundle of (oriented) transverse orthonormal framings. The *transverse spin structure* [3] is a principal Spin(q)-bundle P_{spin} together with two sheeted covering $\xi : P_{\mathrm{spin}} \to P_{\mathrm{so}}$ such that $\xi(p\cdot g) = \xi(p)\xi_0(g)$ for all $p \in P_{\mathrm{spin}}$, $g \in \mathrm{Spin(q)}$, where $\xi_0 : \mathrm{Spin(q)} \to \mathrm{SO(q)}$ is a covering. The Riemannian foliation $\mathcal{F}$ is called a *Kähler spin foliation* if $\mathcal{F}$ is Kähler foliation with a transverse spin structure. The *foliated spinor bundle* $S(\mathcal{F})$ of the Kähler spin foliation $\mathcal{F}$ is defined by

$$S(\mathcal{F}) = P_{\mathrm{spin}} \times_{\mathrm{Spin}(q)} S,$$

where S is the spinor space associated to Q, which is a Clifford module over the transversal Clifford algebra $\mathrm{Cl}(Q)$ of $\mathcal{F}$. The Hermitian scalar product $\langle , \rangle$ defined on S induces a Hermitian scalar product on $S(\mathcal{F})$, which we also denote by $\langle , \rangle$. The sections of $S(\mathcal{F})$ are called *transversal spinor fields*.

By the Clifford multiplication in the fibers of $S(\mathcal{F})$ for any vector field X in Q and any transversal spinor field ψ, the Clifford product $X\cdot\psi$, which is also a transversal spinor field, is defined. This product has the following properties: for all $X, Y \in \Gamma Q$ and $\phi, \psi \in \Gamma S(\mathcal{F})$,

$$(X\cdot Y + Y\cdot X)\psi = -2g_Q(X,Y)\psi, \tag{3.1}$$

$$\langle X\cdot\psi, \phi\rangle + \langle\psi, X\cdot\phi\rangle = 0, \tag{3.2}$$

$$\nabla_Y(X\cdot\psi) = (\nabla_Y X)\cdot\psi + X\cdot(\nabla_Y\psi), \tag{3.3}$$

where ∇ is a metric covariant derivation on $S(\mathcal{F})$, i.e., for all $X \in \Gamma Q$, and all $\psi, \phi \in \Gamma S(\mathcal{F})$, it holds

$$X\langle\psi, \phi\rangle = \langle\nabla_X\psi, \phi\rangle + \langle\psi, \nabla_X\phi\rangle. \tag{3.4}$$

Moreover, if we define the Clifford product $\xi\cdot\psi$ of a 1-form $\xi \in Q^*$ and a transversal spinor field ψ as

$$\xi\cdot\psi \equiv X_\xi\cdot\psi, \tag{3.5}$$

where $X_\xi \in \Gamma Q$ is a g_Q-dual vector of ξ, then any basic r-form can be considered as an endomorphism of $S(\mathcal{F})$. Namely, for any basic form $\omega = \sum_{i_1<\cdots<i_r}\omega_{i_1\cdots i_r}\theta^{i_1}\wedge\cdots\wedge\theta^{i_r}$, we define the Clifford product $\omega\phi$ locally by

$$\omega\phi = \sum\omega_{i_1\cdots i_r}\theta_{i_1}\cdots\theta_{i_r}\phi. \tag{3.6}$$

So for any basic r-form ω, the equation

$$\langle\omega\phi, \psi\rangle = (-1)^{r(r+1)/2}\langle\phi, \omega\psi\rangle \tag{3.7}$$

holds, i.e., the adjoint of ω^* is given by

$$\omega^* = (-1)^{r(r+1)/2}\omega. \tag{3.8}$$

From (2.10) and (3.6), we know that

$$\Omega = -\frac{1}{2}\sum_a E_a\cdot JE_a = \frac{1}{2}\sum_a JE_a\cdot E_a, \tag{3.9}$$

where $\{E_a\}$ is a local orthonormal basic frame in Q. From (3.9), the relation

$$X\cdot\Omega - \Omega\cdot X = 2JX \quad \text{for} \quad X \in \Gamma Q \tag{3.10}$$

holds.

Lemma 3.1 (cf. [5]). *On the Kähler spin foliation, the eigenvalues of* $\Omega_x : S_x(\mathcal{F}) \to S_x(\mathcal{F})$ $(x \in M)$ *are*

$$\mu_r = (n-2r)\mathrm{i}, \quad r = 0, \ldots, n. \tag{3.11}$$

From (3.11), the foliated spinor bundle $S(\mathcal{F})$ of a Kähler spin foliation $\mathcal{F}$ splits into the orthogonal direct sum

$$S(\mathcal{F}) = S_0 \oplus S_1 \oplus \cdots \oplus S_n, \tag{3.12}$$

where the fiber $(S_r)_x$ of the subbundle S_r is just defined as the eigenspace corresponding to the eigenvalue μ_r of $\Omega_x : S_x(\mathcal{F}) \to S_x(\mathcal{F})$. The decomposition (3.12) is compatible with ∇, i.e., if ψ is a section of S_r, then $\nabla_X\psi$ is also a section of S_r for any vector field X.

Let $p_r : S(\mathcal{F}) \to S(\mathcal{F})$ $(r = 0, \dots, n)$ be the projections corresponding to the decomposition (3.12). Then we have the following properties:

$$p_r^2 = p_r, \qquad p_r p_s = p_s p_r = 0, \quad r \neq s, \qquad \sum_{r=0}^{n} p_r = 1, \tag{3.13}$$

$$\langle p_r\psi, \phi\rangle = \langle \psi, p_r\phi\rangle, \qquad \nabla p_r = 0, \qquad S_r = p_r S(\mathcal{F}). \tag{3.14}$$

Hence we get

$$\Omega = \sum_{r=0}^{n} \mathrm{i}(n - 2r) p_r. \tag{3.15}$$

For any vector field $X \in \Gamma Q$, we have the relations

$$X p_s = p_{s-1} X p_s + p_{s+1} X p_s, \tag{3.16}$$

$$J(X) p_s = -\mathrm{i} p_{s-1} X p_s + \mathrm{i} p_{s+1} X p_s, \quad s \in \mathbb{N}, \tag{3.17}$$

where $p_s = 0$ for $s \neq \{0, 1, \dots, n\}$.

Let $\iota : S(\mathcal{F}) \to S(\mathcal{F})$ be the bundle map defined by

$$\iota = \sum_{s=0}^{n} \mathrm{i}^s p_s. \tag{3.18}$$

Then ι has the properties

$$\iota^*\iota = 1, \qquad \iota^2 = \iota^{*2}, \qquad \iota^4 = 1, \qquad \iota^3 = \iota^*, \qquad \nabla\iota = 0. \tag{3.19}$$

For any vector field $X \in \Gamma Q$, the equations

$$J(X)\iota = \iota X, \qquad X\iota^2 = -\iota^2 X \tag{3.20}$$

are satisfied. The proofs of the above equations are similar to the usual ones in Kähler geometry [5].

4. The transversal Dirac operators

Let $\mathcal{F}$ be a Kähler spin foliation on a compact oriented manifold M. Then the transversal Dirac operator $D_{\mathrm{tr}} : \Gamma S(\mathcal{F}) \to \Gamma S(\mathcal{F})$ is locally given by [1–3]

$$D_{\mathrm{tr}}\phi = \sum_a E_a \cdot \nabla_{E_a}\phi - \frac{1}{2}\kappa \cdot \phi \quad \text{for } \phi \in \Gamma S(\mathcal{F}), \tag{4.1}$$

where $\{E_a\}_{a=1,\dots,2n}$ is a local orthonormal basic frame in Q. Let $\tilde{D}_{\mathrm{tr}}$ be the operator which is locally defined by

$$\tilde{D}_{\mathrm{tr}}\phi = \sum_a JE_a \cdot \nabla_{E_a}\phi - \frac{1}{2}J\kappa \cdot \phi \quad \text{for } \phi \in \Gamma S(\mathcal{F}). \tag{4.2}$$

Using Green's theorem on the foliated Riemannian manifold [9], we know for any $\phi, \psi \in \Gamma S(\mathcal{F})$

$$\int_M \langle D_{\rm tr}\phi, \psi\rangle = \int_M \langle \phi, D_{\rm tr}\psi\rangle, \qquad \int_M \langle \tilde{D}_{\rm tr}\phi, \psi\rangle = \int_M \langle \phi, \tilde{D}_{\rm tr}\psi\rangle, \tag{4.3}$$

i.e., $D_{\rm tr}$ and $\tilde{D}_{\rm tr}$ are self-adjoint transversally elliptic operators. From $\nabla\iota = 0$ and (3.20), we obtain

$$\tilde{D}_{\rm tr}\iota = \iota D_{\rm tr}, \qquad \iota\tilde{D}_{\rm tr} = -D_{\rm tr}\iota. \tag{4.4}$$

From (4.4), we get

$$D_{\rm tr}\iota^2 = -\iota^2 D_{\rm tr}, \qquad \tilde{D}_{\rm tr}\iota^2 = -\iota^2\tilde{D}_{\rm tr}. \tag{4.5}$$

From (4.4) and (4.5), we have

$$D_{\rm tr}^2\iota = \iota\tilde{D}_{\rm tr}^2, \qquad D_{\rm tr}\iota D_{\rm tr}\iota = \iota\tilde{D}_{\rm tr}\iota\tilde{D}_{\rm tr}. \tag{4.6}$$

Moreover, from (3.16) and (3.17) and their Hermitian adjoint equations, we have

$$p_s\tilde{D}_{\rm tr} - \tilde{D}_{\rm tr}p_{s-1} = -{\rm i}(p_s D_{\rm tr} - D_{\rm tr}p_{s-1}), \quad s \in \mathbb{N}. \tag{4.7}$$

We now define $\nabla_{\rm tr}^*\nabla : \Gamma S(\mathcal{F}) \to \Gamma S(\mathcal{F})$ as

$$\nabla_{\rm tr}^*\nabla_{\rm tr}\phi = -\sum_a \nabla^2_{E_a,E_a}\phi + \nabla_\kappa\phi, \tag{4.8}$$

where $\nabla^2_{V,W} = \nabla_V\nabla_W - \nabla_{\nabla_V W}$ for $V, W \in TM$. Then we have the following proposition.

Proposition 4.1. (see [3]). *For all* $\phi, \psi \in \Gamma S(\mathcal{F})$,

$$\int_M \langle \nabla_{\rm tr}^*\nabla_{\rm tr}\phi, \psi\rangle = \int_M \langle \nabla_{\rm tr}\phi, \nabla_{\rm tr}\psi\rangle. \tag{4.9}$$

If $\mathcal{F}$ is isoparametric, i.e., $\kappa \in \Omega_B^1$, then we have [3]

$$D_{\rm tr}^2\phi = \nabla_{\rm tr}^*\nabla_{\rm tr}\phi + \tfrac{1}{4}\sigma_\nabla\phi + K_\nabla\phi, \tag{4.10}$$

where $K_\nabla = (1/2)\{-\delta\kappa + (1/2)|\kappa|^2\}$. By direct calculation, we also have

$$\tilde{D}_{\rm tr}^2\phi = \nabla_{\rm tr}^*\nabla_{\rm tr}\phi + \frac{1}{4}\sigma_\nabla\phi - \frac{1}{4}|\kappa|^2\phi - \frac{1}{2}\sum_a JE_a\cdot J(\nabla_{E_a}\kappa)\cdot\phi. \tag{4.11}$$

If κ is a transversally holomorphic (see (2.16)), we have, from the definition of Clifford multiplication and (2.6),

$$\sum_a JE_a\cdot J(\nabla_{E_a}\kappa) = \sum_a JE_a\cdot\nabla_{JE_a}\kappa = d_B\kappa + \delta_B\kappa - |\kappa|^2.$$

If $\mathcal{F}$ is an isoparametric, κ is already closed, i.e., ${\rm d}\kappa = 0$ [8]. So we have

$$\tilde{D}_{\rm tr}^2\phi = \nabla_{\rm tr}^*\nabla_{\rm tr}\phi + \tfrac{1}{4}\sigma_\nabla\phi + K_\nabla\phi. \tag{4.12}$$

Then we have the following proposition.

Proposition 4.2. *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a Kähler spin foliation $\mathcal{F}$ and a bundle-like metric g_M with $\kappa \in \Omega_B^1(\mathcal{F})$. Suppose the mean curvature of $\mathcal{F}$ is a transversally holomorphic. Then we have*

$$D_{\mathrm{tr}}^2 = \tilde{D}_{\mathrm{tr}}^2, \qquad D_{\mathrm{tr}}\tilde{D}_{\mathrm{tr}} + \tilde{D}_{\mathrm{tr}}D_{\mathrm{tr}} = 0.$$

Proof. The first equation is trivial from (4.10) and (4.12). Next, fix $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ with the property that $(\nabla E_a)_x = 0$ for all a. Then we have at the point x that for any $\phi \in \Gamma S(\mathcal{F})$,

$$\begin{aligned}
D_{\mathrm{tr}}\tilde{D}_{\mathrm{tr}}\phi &= D_{\mathrm{tr}}\left(\sum_a JE_a\cdot\nabla_{E_a}\phi - \frac{1}{2}J\kappa\cdot\phi\right)\\
&= \sum_{a,b} E_b\cdot\nabla_{E_b}(JE_a\cdot\nabla_{E_a}\phi) - \frac{1}{2}\kappa\cdot JE_a\cdot\nabla_{E_a}\phi - \frac{1}{2}D_{\mathrm{tr}}(J\kappa\cdot\phi)\\
&= \sum_{a,b} E_b\cdot JE_a\cdot\nabla_{E_b}\nabla_{E_a}\phi - \frac{1}{2}\kappa\cdot JE_a\cdot\nabla_{E_a}\phi\\
&\quad -\frac{1}{2}\left\{d_B(J\kappa) + \delta_B(J\kappa) + \sum_a E_a\cdot J\kappa\cdot\nabla_{E_a}\phi - \frac{1}{2}\kappa\cdot J\kappa\cdot\phi\right\}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\tilde{D}_{\mathrm{tr}}D_{\mathrm{tr}} &= \sum_{a,b} JE_a\cdot E_b\cdot\nabla_{E_a}\nabla_{E_b}\phi - \frac{1}{2}J\kappa\cdot E_a\cdot\nabla_{E_a}\phi\\
&\quad -\frac{1}{2}\left\{-d_B(J\kappa) - \delta_B(J\kappa) + \sum_a JE_a\cdot\kappa\cdot\nabla_{E_a}\phi - \frac{1}{2}J\kappa\cdot\kappa\cdot\phi\right\}.
\end{aligned}$$

Since $X\cdot Y + Y\cdot X = -2g_Q(X, Y)$ and g_Q is Hermitian, we have

$$\begin{aligned}
(D_{\mathrm{tr}}\tilde{D}_{\mathrm{tr}} + \tilde{D}_{\mathrm{tr}}D_{\mathrm{tr}})\phi &= \sum_{a,b}(E_b\cdot JE_a + JE_b\cdot E_a)\nabla_{E_b}\nabla_{E_a}\phi\\
&= \sum_{a,b}(JE_b\cdot JE_a - E_b\cdot E_a)\nabla_{JE_b}\nabla_{E_a}\phi\\
&= \sum_{a,b}(-JE_a\cdot JE_b + E_a\cdot E_b)\nabla_{JE_b}\nabla_{E_a}\phi\\
&= \sum_{a,b}(-JE_a\cdot JE_b + E_a\cdot E_b)(\nabla_{E_a}\nabla_{JE_b}\phi + R^S(JE_b, E_a)\phi)\\
&= -\sum_{a,b}(JE_a\cdot E_b + E_a\cdot JE_b)\nabla_{E_a}\nabla_{E_b}\phi\\
&\quad - \sum_{a,b}(JE_a\cdot E_b + E_a\cdot JE_b)R^S(E_b, E_a)\phi\\
&= -(D_{\mathrm{tr}}\tilde{D}_{\mathrm{tr}} + \tilde{D}_{\mathrm{tr}}D_{\mathrm{tr}})\phi.
\end{aligned}$$

This finishes the proof. □

From (4.6), we have the following corollary.

Corollary 4.3. *On an isoparametric Kähler spin foliation $\mathcal{F}$ with a transversally holomorphic mean curvature κ, we have*

$$D_{\rm tr}^2\iota = \iota D_{\rm tr}^2, \qquad D_{\rm tr}\iota D_{\rm tr}\iota = \iota D_{\rm tr}\iota D_{\rm tr}.$$

5. Eigenvalue estimate

Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a Kähler spin foliation $\mathcal{F}$ of codimension $q = 2n$ and a bundle-like metric g_M with respect to $\mathcal{F}$. On the foliated spinor bundle $S(\mathcal{F})$, we introduce a new connection of the form

$$\overset{fg}{\nabla}_X\phi = \nabla_X\phi + f\pi(X)\cdot\phi + {\rm i}gJ\pi(X)\cdot\iota^2\phi \quad \text{for} \quad X \in TM, \tag{5.1}$$

where f, g are real valued basic functions on M and $\pi : TM \to Q$. Trivially, this connection ∇^{fg} is a metrical connection. Moreover, we have the following lemma.

Lemma 5.1. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a Kähler foliation $\mathcal{F}$ and a bundle-like metric g_M with respect to $\mathcal{F}$. Then,*

$$\langle\langle \overset{fg}{\nabla}{}^*_{\rm tr}\overset{fg}{\nabla}_{\rm tr}\phi, \psi\rangle\rangle = \langle\langle \overset{fg}{\nabla}_{\rm tr}\phi, \overset{fg}{\nabla}_{\rm tr}\psi\rangle\rangle$$

for all $\phi, \psi \in \Gamma S$, where $\langle\langle\phi, \psi\rangle\rangle = \int_M \langle\phi, \psi\rangle$ is the Hermitian inner product on $S(\mathcal{F})$.

Proof. Fix $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ such that $(\nabla E_a)_x = 0$ for all a. Then we have that at x,

$$\begin{aligned}\langle \overset{fg}{\nabla}{}^*_{\rm tr}\overset{fg}{\nabla}_{\rm tr}\phi, \psi\rangle &= -\sum_a \langle \overset{fg}{\nabla}_{E_a}\overset{fg}{\nabla}_{E_a}\phi, \psi\rangle + \langle \overset{fg}{\nabla}_\kappa\phi, \psi\rangle = -\sum_a E_a\langle \overset{fg}{\nabla}_{E_a}\phi, \psi\rangle \\ &\quad + \sum_a \langle \overset{fg}{\nabla}_{E_a}\phi, \overset{fg}{\nabla}_{E_a}\psi\rangle + \langle \overset{fg}{\nabla}_\kappa\phi, \psi\rangle = -\sum_a E_a\langle \nabla_{E_a}\phi, \psi\rangle \\ &\quad - \sum_a E_a\langle fE_a\phi, \psi\rangle - \sum_a E_a\langle {\rm i}gJE_a\cdot\iota^2\phi, \psi\rangle + \sum_a \langle \overset{fg}{\nabla}_{E_a}\phi, \overset{fg}{\nabla}_{E_a}\psi\rangle \\ &\quad + \langle\nabla_\kappa\phi, \psi\rangle + \langle f\kappa\cdot\phi, \psi\rangle + \langle {\rm i}gJ\kappa\cdot\iota^2\phi, \psi\rangle = -{\rm div}_\nabla U \\ &\quad - {\rm div}_\nabla V - {\rm div}_\nabla W + \sum_a \langle \overset{fg}{\nabla}_{E_a}\phi, \overset{fg}{\nabla}_{E_a}\psi\rangle + \langle\nabla_\kappa\phi, \psi\rangle \\ &\quad + \langle f\kappa\cdot\phi, \psi\rangle + \langle {\rm i}gJ\kappa\cdot\iota^2\phi, \psi\rangle,\end{aligned}$$

where $U, V, W \in \Gamma Q \otimes \mathbb{C}$ are defined by the conditions that $g_Q(U, Z) = \langle\nabla_Z\phi, \psi\rangle$, $g_Q(fV, Z) = \langle fZ\cdot\phi, \psi\rangle$ and $g_Q(gW, Z) = \langle gJZ\cdot\iota^2\phi, \psi\rangle$ for all $Z \in \Gamma Q$. The last line is proved as follows. At $x \in M$,

$${\rm div}_\nabla(U) = \sum_a g_Q(\nabla_{E_a}U, E_a) = \sum_a E_a g_Q(U, E_a) = \sum_a E_a\langle\nabla_{E_a}\phi, \psi\rangle.$$

Similarly, we have that

$$\mathrm{div}_\nabla(fV) = \sum_a E_a\langle fE_a\cdot\phi,\psi\rangle,\qquad \mathrm{div}_\nabla(gW) = \sum_a E_a\langle gJE_a\cdot\iota^2\phi,\psi\rangle.$$

By the Green's theorem on the foliated Riemannian manifold [9],

$$\int_M \mathrm{div}_\nabla(V) = \langle\langle\kappa, V\rangle\rangle = \langle\langle\nabla_\kappa\phi,\psi\rangle\rangle. \tag{5.2}$$

Similarly, we have

$$\int_M \mathrm{div}_\nabla(fV) = \langle\langle f\kappa\cdot\phi,\psi\rangle\rangle,\qquad \int_M \mathrm{div}_\nabla(gW) = \langle\langle gJ\kappa\cdot\iota^2\phi,\psi\rangle\rangle.$$

By integrating, we obtain our result. □

On the other hand, by direct calculation, we have

$$\begin{aligned}\overset{fg}{\nabla}{}^*_{\mathrm{tr}}\overset{fg}{\nabla}_{\mathrm{tr}}\phi &= -\sum_a \overset{fg}{\nabla}_{E_a}\overset{fg}{\nabla}_{E_a}\phi + \overset{fg}{\nabla}_\kappa\phi = -\sum_a \nabla_{E_a}\nabla_{E_a}\phi + \nabla_\kappa\phi\\ &\quad -2f\sum_a E_a\cdot\nabla_{E_a}\phi + 2\mathrm{i}g\iota^2\sum_a JE_a\cdot\nabla_{E_a}\phi - f^2\sum_a E_a\cdot E_a\cdot\phi\\ &\quad + g^2\sum_a JE_a\iota^2 JE_a\iota^2\phi - 2\mathrm{i}fg\sum_a E_a\cdot JE_a\iota^2\phi - \sum_a E_a(f)E_a\phi\\ &\quad -\mathrm{i}\sum_a E_a(g)JE_a\iota^2\phi + f\kappa\cdot\phi + \mathrm{i}gJ\kappa\cdot\iota^2\phi.\end{aligned}$$

From this equation, we obtain

$$\begin{aligned}\overset{fg}{\nabla}{}^*_{\mathrm{tr}}\overset{fg}{\nabla}_{\mathrm{tr}}\phi &= \nabla^*_{\mathrm{tr}}\nabla_{\mathrm{tr}}\phi - 2fD_{\mathrm{tr}}\phi + 2\mathrm{i}g\iota^2\tilde{D}_{\mathrm{tr}}\phi + q(f^2+g^2)\phi\\ &\quad + 4\mathrm{i}fg\Omega\iota^2\phi - \mathrm{grad}_\nabla(f)\cdot\phi - \mathrm{i}J(\mathrm{grad}_\nabla(g))\cdot\iota^2\phi,\end{aligned}$$

where $\mathrm{grad}_\nabla(f) = \sum_a E_a(g)E_a$ is a transversal gradient of f. From (4.10), we get

$$\begin{aligned}\overset{fg}{\nabla}{}^*_{\mathrm{tr}}\overset{fg}{\nabla}_{\mathrm{tr}}\phi &= D^2_{\mathrm{tr}}\phi - 2fD_{\mathrm{tr}}\phi + 2\mathrm{i}g\iota^2\tilde{D}_{\mathrm{tr}}\phi + 4\mathrm{i}fg\Omega\iota^2\phi - \mathrm{grad}_\nabla(f)\cdot\phi\\ &\quad - \mathrm{i}J(\mathrm{grad}_\nabla(g))\cdot\iota^2\phi + \{q(f^2+g^2) - \tfrac{1}{4}\sigma_\nabla - K_\nabla\}\phi.\end{aligned} \tag{5.3}$$

Let $E_\lambda(D_{\mathrm{tr}})$ be the eigenspace of the transversal Dirac operator D_{tr} corresponding to the eigenvalue λ. Let $\phi\in E_\lambda(D_{\mathrm{tr}})$. Then we have

$$\begin{aligned}\|\overset{fg}{\nabla}_{\mathrm{tr}}\phi\|^2 &= \lambda^2\|\phi\|^2 - 2f\lambda\|\phi\|^2 + 2g\langle\langle\mathrm{i}\tilde{D}_{\mathrm{tr}}\phi,\iota^2\phi\rangle\rangle + 4fg\langle\langle\mathrm{i}\Omega\phi,\iota^2\phi\rangle\rangle\\ &\quad - \langle\langle\mathrm{grad}_\nabla(f)\phi,\phi\rangle\rangle - \mathrm{i}\langle\langle J(\mathrm{grad}_\nabla(g))\iota^2\phi,\phi\rangle\rangle\\ &\quad + \{q(f^2+g^2) - \tfrac{1}{4}\sigma_\nabla - K_\nabla\}\|\phi\|^2.\end{aligned} \tag{5.4}$$

From (4.4)–(4.6), we have the following lemma.

Lemma 5.2 (cf. [5]). *Let* $\phi \in E_\lambda(D_{\rm tr})$. *Then* $f_\lambda : E_\lambda(D_{\rm tr}) \to E_\lambda(D_{\rm tr})$ *defined by*

$$f_\lambda(\phi) = (D_{\rm tr} + \lambda)\iota^*\phi$$

satisfies

$$f_\lambda^4 + 4\lambda^4 = 0. \tag{5.5}$$

The above equation shows that the eigenspace $E_\lambda(D_{\rm tr})$ is decomposed as

$$E_\lambda(D_{\rm tr}) = \oplus_{\ell=0}^3 E_\lambda^\ell(D_{\rm tr}), \tag{5.6}$$

where $E_\lambda^\ell(D_{\rm tr}) = \{\phi \in E_\lambda(D_{\rm tr}) | f_\lambda\phi = {\rm i}^\ell(1+{\rm i})\lambda\phi\}$ $(\ell = 0, 1, 2, 3)$. A corollary of Lemma 5.2 is the following proposition.

Proposition 5.3 (cf. [5]). *For any nonzero* $\phi \in E_\lambda^\ell(D_{\rm tr})$, *we have*

$$\tilde{D}_{\rm tr}\phi = \lambda({\rm i}^\ell(1+{\rm i})\iota - 1)\phi. \tag{5.7}$$

From (4.7) and (5.7), we have the following proposition.

Proposition 5.4 (cf. [5]). *For any nonzero* $\phi \in E_\lambda^\ell(D_{\rm tr})$, *we have*

$$\|p_{4s-\ell-1}\phi\| = \|p_{4s-\ell}\phi\|, \qquad p_{4s-\ell+1}\phi = p_{4s-\ell+2}\phi = 0, \qquad s \in \mathbb{N} \cup \{0\}. \tag{5.8}$$

From (3.15), (5.7) and (5.8), we have the following corollary.

Corollary 5.5 (cf. [5]). *For* $\phi \in E_\lambda^\ell(D_{\rm tr})$,

$$\langle\langle {\rm i}\tilde{D}_{\rm tr}\phi, \iota^2\phi\rangle\rangle = (-1)^{\ell+1}\lambda\|\phi\|^2, \qquad \langle\langle {\rm i}\Omega\phi, \iota^2\phi\rangle\rangle = (-1)^\ell\|\phi\|^2.$$

Note that for all $X \in \Gamma Q$ and $\phi \in \Gamma S$,

$$\langle X\cdot\phi, \phi\rangle = \overline{\langle\phi, X\cdot\phi\rangle} = -\overline{\langle X\cdot\phi, \phi\rangle}, \tag{5.9}$$

$$\langle JX\iota^2\phi, \phi\rangle = \overline{\langle\phi, JX\iota^2\phi\rangle} = -\overline{\langle JX\cdot\phi, \iota^2\phi\rangle} = -\overline{\langle\iota^2 JX\cdot\phi, \phi\rangle} = \overline{\langle JX\iota^2\phi, \phi\rangle}. \tag{5.10}$$

So we know that $\langle {\rm grad}_\nabla(f)\phi, \phi\rangle$ and ${\rm i}\langle J({\rm grad}_\nabla(g))\iota^2\phi, \phi\rangle$ are purely imaginary. Hence if we combine (5.4) and Corollary 5.5, then we have, for $\phi \in E_\lambda^\ell(D_{\rm tr})$,

$$\|\overset{fg}{\nabla}_{\rm tr}\phi\|^2 = \int_M \left(F(x,y)\lambda^2 - \frac{1}{4}\sigma_\nabla - K_\nabla\right)|\phi|^2, \tag{5.11}$$

$$\langle\langle {\rm grad}_\nabla(f)\phi, \phi\rangle\rangle + {\rm i}\langle\langle J({\rm grad}_\nabla(g))\iota^2\phi, \phi\rangle\rangle = 0, \tag{5.12}$$

where $f = \lambda x$, $g = \lambda y$ and $F(x,y) = qx^2 + qy^2 + 4(-1)^\ell xy - 2x - 2(-1)^\ell y + 1$.

It is straightforward to notice the following lemma.

Lemma 5.6. *The polynomial* F *has its minimum* $q/(q+2)$ *at the point* $(1/(q+2), (-1)^\ell/(q+2))$.

Now we assume that the mean curvature κ of $\mathcal{F}$ satisfies $\delta\kappa = 0$. And if we put $f = \lambda/(q+2)$ and $g = (-1)^\ell \lambda/(q+2)$, then (5.11) takes the form

$$\|\overset{fg}{\nabla}_{\mathrm{tr}}\phi\|^2 = \int_M \left(\frac{q}{q+2}\lambda^2 - \frac{1}{4}K_\sigma\right)|\phi|^2, \tag{5.13}$$

where $K_\sigma = \sigma^\nabla + |\kappa|^2$. From (5.13), we have the following theorem.

Theorem 5.7. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a Kähler spin foliation $\mathcal{F}$ of codimension $q = 2n$ and a bundle-like metric g_M with $\kappa \in \Omega_B^1(\mathcal{F})$. Assume that the mean curvature κ of $\mathcal{F}$ satisfies $\delta\kappa = 0$ and transversally holomorphic. If $K_\sigma \geq 0$, then the eigenvalue λ of D_{tr} satisfies*

$$\lambda^2 \geq \frac{q+2}{4q}K_\sigma^0,$$

where $K_\sigma^0 = \min K_\sigma$.

Remark 5.8. If $\mathcal{F}$ is a point foliation, then the transversal Dirac operator is just a Dirac operator on a Kähler manifold. Therefore, Theorem 5.7 is a generalization of the result on a Kähler manifold (cf. [5]).

Remark 5.9. The estimation of the eigenvalue of the transversal Dirac operator on a Kähler spin foliation is a sharper one than the estimation (1.1).

6. The limiting case

In this section, we study the Kähler spin foliation which admits a nonzero transversal spinor ϕ_1 such that $D_{\mathrm{tr}}\phi_1 = \lambda_1\phi_1$ with $\lambda_1 = ((q+2)/4q)K_\sigma^0$. From (5.13), we have that for any $\phi_1 \in E_{\lambda_1}^\ell(D_{\mathrm{tr}})$

$$\|\nabla_{\mathrm{tr}}^f \phi_1\|^2 = \int_M \frac{1}{4}(K_\sigma^0 - K_\sigma)|\phi_1|^2, \tag{6.1}$$

where

$$\nabla_X^f \phi = \nabla_X\phi + f\pi(X)\phi + \mathrm{i}(-1)^\ell fJ\pi(X)\iota^2\phi. \tag{6.2}$$

From this equation, we have

$$K_\sigma = K_\sigma^0 \quad \text{and} \quad \nabla_{\mathrm{tr}}^f\phi_1 = 0. \tag{6.3}$$

From the first equation in (6.3), if the transversal scalar curvature σ_∇ is nonnegative, we know that

$$\sigma_\nabla = \text{constant} \quad \text{and} \quad |\kappa| = \text{constant}. \tag{6.4}$$

From the second equation in (6.3), we have

$$\sum_a E_a \cdot \nabla_{E_a}\phi_1 + f\sum_a E_a \cdot E_a \cdot \phi_1 + \mathrm{i}(-1)^\ell f \sum_a E_a \cdot JE_a\iota^2\phi_1 = 0,$$

where $\{E_a\}$ is an orthonormal basic frame on Q. From this equation, we have

$$D_{\rm tr}\phi_1 + \tfrac{1}{2}\kappa\cdot\phi_1 - qf\phi_1 - 2{\rm i}(-1)^{\ell}\Omega\iota^2\phi_1 = 0.$$

Since $D_{\rm tr}\phi_1 = \lambda_1\phi_1$ and $f = \lambda_1/(q+2)$, we have

$$(1 - {\rm i}(-1)^{\ell}\Omega\iota^2)\phi_1 = -\frac{1}{4f}\kappa\cdot\phi_1. \tag{6.5}$$

From the second equation in (6.3), we also have

$$\sum_a JE_a\cdot\nabla_{E_a}\phi_1 + f\sum_a JE_a\cdot E_a\cdot\phi_1 + {\rm i}(-1)^{\ell}f\sum_a JE_a\cdot JE_a\iota^2\phi_1 = 0.$$

This equation implies that

$$\tilde{D}_{\rm tr}\phi_1 + \tfrac{1}{2}J\kappa\cdot\phi_1 + 2f\Omega\phi_1 - {\rm i}(-1)^{\ell}qf\iota^2\phi_1 = 0.$$

From (5.7), we have

$$(q+2)f({\rm i}^{\ell}(1+{\rm i})\iota - 1)\phi_1 + \tfrac{1}{2}J\kappa\cdot\phi_1 + 2f\Omega\phi_1 - {\rm i}(-1)^{\ell}qf\iota^2\phi_1 = 0. \tag{6.6}$$

By applying ι^2 to (6.7) and using (3.20), we get

$$(q+2)f({\rm i}^{\ell}(1+{\rm i})\iota - 1)\iota^2\phi_1 - \tfrac{1}{2}J\kappa\iota^2\phi_1 + 2f\Omega\iota^2\phi_1 - {\rm i}(-1)^{\ell}qf\phi_1 = 0. \tag{6.7}$$

Hence this equation is equivalent to

$$(1 - {\rm i}(-1)^{\ell}\Omega\iota^2)\phi_1 = \frac{q+2}{2}\{1 - {\rm i}(-1)^{\ell}(1 - {\rm i}^{\ell}(1+{\rm i})\iota)\iota^2\}\phi_1 - {\rm i}\frac{(-1)^{\ell}}{4f}J\kappa\iota^2\phi_1. \tag{6.8}$$

From (5.8), we obtain

$$\{1 - {\rm i}(-1)^{\ell}(1 - {\rm i}^{\ell}(1+{\rm i})\iota)\iota^2\}\phi_1 = 0.$$

Hence the formula (6.8) is equivalent to

$$(1 - {\rm i}(-1)^{\ell}\Omega\iota^2)\phi_1 = -{\rm i}\frac{(-1)^{\ell}}{4f}J\kappa\cdot\iota^2\phi_1. \tag{6.9}$$

Combining (6.5) with (6.9), we have

$$J\kappa\cdot\phi_1 = {\rm i}(-1)^{\ell+1}\kappa\cdot\phi_1. \tag{6.10}$$

By long calculation, we have that for $X \in \Gamma Q$ and $\phi \in E_{\lambda}^{\ell}(D_{\rm tr})$

$$\sum_a E_a\cdot R^f_{XE_a}\phi = \sum_a E_a\cdot R^S_{XE_a}\phi + 2(q+2)f^2X + 4{\rm i}(-1)^{\ell}f^2JX\iota^2(1 - {\rm i}(-1)^{\ell}\Omega\iota^2)\phi, \tag{6.11}$$

where R^f is a curvature tensor of ∇^f and R^S a curvature tensor of ∇ on $S(\mathcal{F})$ which is given by [6]

$$R^S_{XY}\phi = \frac{1}{4}\sum_{a,b} g_Q(R^{\nabla}_{XY}E_a, E_b)E_a\cdot E_b\cdot\phi \quad \text{for} \ X, Y\in TM.$$

If $\nabla^f\phi = 0$, then $R^f_{XY}\phi = 0$. Hence we have that for any $\phi\in E^{\ell}_{\lambda}(D_{\mathrm{tr}})$

$$\sum_a E_a\cdot R^S(X, E_a)\phi = -f^2\{2(q+2)X + 4\mathrm{i}(-1)^{\ell}JX\iota^2(1-\mathrm{i}(-1)^{\ell}\Omega\iota^2)\}\phi, \quad (6.12)$$

where $\{E_a\}_{a=1,\dots,q}$ is an orthonormal basic frame of Q.

If we substitute (6.9) into (6.12), then we get that for any $\phi\in E^{\ell}_{\lambda}(D_{\mathrm{tr}})$

$$\sum_a E_a\cdot R^S(X, E_a)\phi = -f^2\left\{2(q+2)X - \frac{1}{f}JX\cdot J\kappa\right\}\phi. \quad (6.13)$$

On the foliated spinor bundle $S(\mathcal{F})$, we have [6] that for any $\phi\in E^{\ell}_{\lambda}(D_{\mathrm{tr}})$

$$\sum_a E_aR^S(X, E_a)\phi = -\tfrac{1}{2}\rho^{\nabla}(X)\cdot\phi \quad \text{for} \ X\in\Gamma Q. \quad (6.14)$$

If we compare (6.13) with (6.14), then we obtain

$$\rho^{\nabla}(X) = 4f^2(q+2)X - 2fJX\cdot J\kappa \quad \text{for} \ X\in\Gamma Q. \quad (6.15)$$

From (6.15), we have

$$\langle\rho^{\nabla}(\kappa)\cdot\phi,\phi\rangle = 4f^2(q+2)\langle\kappa\cdot\phi,\phi\rangle - 2f|\kappa|^2\langle\phi,\phi\rangle.$$

From (5.9), the left-hand side is purely imaginary. Hence we have

$$|\kappa|^2\langle\phi,\phi\rangle = 0. \quad (6.16)$$

Because $\phi\neq 0$ at some point $x\in M$, this implies that $|\kappa|(x) = 0$ and then from (6.4), $|\kappa| = 0$ for any $x\in M$. That is, the foliation $\mathcal{F}$ is minimal. So (6.15) implies that

$$\rho^{\nabla}(X) = 4f^2(q+2)X \quad \text{for} \ X\in\Gamma Q. \quad (6.17)$$

This implies that the $\mathcal{F}$ is a transversally Einsteinian.

On the other hand, since $\mathcal{F}$ is minimal, from (6.9), we have

$$(1-\mathrm{i}(-1)^{\ell}\Omega\iota^2)\phi_1 = 0. \quad (6.18)$$

From the definition (3.18) of ι and (3.15), we have

$$0 = (1-\mathrm{i}(-1)^{\ell}\Omega\iota^2)\phi_1 = \sum_{s=0}^{n}(1+(-1)^{\ell+s}(n-2s)p_s)\phi_1. \quad (6.19)$$

Hence from Proposition 5.4, (6.19) is equivalent to

$$\sum_s(n-8s+2\ell+1)(p_{4s-\ell}-p_{4s-\ell-1})\phi_1 = 0. \quad (6.20)$$

If we choose $s \in \mathbb{N}$ such that $p_{4s-\ell}\phi \neq 0$, then $n = 8s + 2\ell + 1$. This imply that n must be odd. So we have the following theorem.

Theorem 6.1. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a Kähler spin foliation $\mathcal{F}$ of codimension $q = 2n$ and a bundle-like metric g_M with $\kappa \in \Omega_B^1(\mathcal{F})$. Assume that the mean curvature κ of $\mathcal{F}$ satisfies $\delta\kappa = 0$ and transversally holomorphic. If there exists an eigenspinor field $\phi(\neq 0)$ of transversal Dirac operator D_{tr} for the eigenvalue $\lambda^2 = ((q+2)/4q)K_\sigma^0$, then $\mathcal{F}$ is a minimal, transversally Einsteinian of odd complex codimension n with nonnegative constant transversal scalar curvature σ^∇.*

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